Fermat’s “infinite descent”

“There is no right triangle with integer side lengths and square area.”

**Proof** Let \(x, y, z \in \mathbb{N}\) be the sides of a right triangle with hypotenuse \(z\) and square area, which means that \[ E = \frac{1}{2}xy \] is a square number. Then necessarily one of the legs \(x, y\) is odd and the other is even while the hypotenuse \(z\) is odd. Let’s assume that the triple \(x, y, z\) is primitive, i.e. \(x, y, z\) are pairwise coprime (in different case we could have divided by their greatest common divisor). As \(x, y, z\) constitute a Pythagorean triple, it follows that (using Euclid’s formulae for Pythagorean triples): \[ x = 2ab, \ y = a^2 - b^2, \ z = a^2 + b^2 \] where \(a, b \in \mathbb{N}, a > b\) and since the triple is primitive, \(a, b\) are coprime and \(a - b\) is odd, which means that \(a, b\) have different parities (one is odd and one is even). Then \[ E = \frac{1}{2} \cdot 2ab \cdot (a^2 - b^2) = ab \cdot (a^2 - b^2) = ab(a - b)(a + b) \] As \(a, b, a - b, a + b\) are pairwise coprime and \(E\) is a square, it follows that \(a, b, a - b, a + b\) are all squares. Let \[ a + b = c^2, \ a - b = d^2 \] where \(c, d \in \mathbb{N}\) and \(c^2, d^2\) are both odd (because one of \(a, b\) is odd and the other is even) therefore \(c, d\) are necessarily odd too. Then \(c - d, c + d\) are both even and one of them is divisible by 4 because if \(d = 2k + 1\), where \(k \in \mathbb{N}\), then their difference is \(c + d - c + d = 2d = 2(2k + 1) = 4k + 2\). Let \[ e = \frac{c - d}{2}, \ f = \frac{c + d}{2} \] one of which is necessarily even. Then \[ e^2 + f^2 = \left(\frac{c - d}{2}\right)^2 + \left(\frac{c + d}{2}\right)^2 = \frac{c^2 + d^2}{2} = \frac{2a}{2} = a \] which is a square. Therefore \(e, f\) are the legs of another right triangle with area \[ \varepsilon = \frac{ef}{2} = \frac{c^2 - d^2}{8} = \frac{2b}{8} = \frac{b}{4} \] But \(b\) is a square therefore \(\varepsilon\) is also a square. Therefore the right triangle with legs \(e, f\) has also a square area which means that if a right triangle has a square area then there necessarily exists another right triangle with a smaller yet still square area and for this still another etc. This is a contradiction because the natural numbers possess a least element.